

HOMOTOPY GROUPS AS CENTERS OF FINITELY PRESENTED GROUPS

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ABSTRACT. For every finite abelian group A and $n \geq 3$, we construct a finitely presented group defined by explicit generators and relations, such that its center is $\pi_n(\Sigma K(A, 1))$.

1. INTRODUCTION

It is shown in [10] that all homotopy groups of spheres and Moore spaces can be presented as centers of certain finitely generated groups given by explicit generators and relations. The following question rises naturally: for which space X and $n \geq 2$, can one construct a finitely presented group $\Gamma(X)$ defined by generators and relations, such that the center of $\Gamma(X)$ is $\pi_n(X)$? In this paper we study this question for suspensions of classifying spaces of finite abelian groups. For every finite abelian group A and $n \geq 2$, we construct a finitely presented group \mathcal{J}_n , such that¹ $Z(\mathcal{J}_n) \simeq \pi_{n+1}(\Sigma K(A, 1))$.

The main approach of the construction of finitely generated groups with centers given by homotopy groups, used in [16] and [10] is the following. For certain simply-connected spaces X there are simplicial group models G_* , for loops of X , i.e. $|G_*| \simeq \Omega X$ such that the centers of components of G_* are trivial and there is a combinatorial description of Moore boundaries $\mathcal{B}G_*$. In this case, the homotopy groups $\pi_{n+1}(X) \simeq \pi_n(G_*)$ are isomorphic to the centers of the quotient groups $G_n/\mathcal{B}G_n$. However, for all such simplicial models in [16] and [10], the Moore boundaries $\mathcal{B}G_n$ are not generated by finitely many elements as normal subgroups of G_n . Recall that, for the two-dimensional sphere S^2 , there is a trick based on properties of braid groups, which gives a sequence of finitely presented groups given by generators and relations such that their centers are homotopy groups $\pi_*(S^2) \times \mathbb{Z}$ [7]. However, this trick does not work for other spaces, since it is based on very specific properties of Milnor's simplicial construction $F[S^1]$ [12]. Recall also that, for a group G , such that the commutator subgroup $[G, G]$ has a trivial center, the non-abelian tensor square $G \otimes G$ in the sense of Brown-Loday [2] has the following property:

$$\pi_3(\Sigma K(G, 1)) \simeq Z(G \otimes G).$$

However, a generalization of this construction for higher homotopy groups seems to be a hard problem.

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¹For a group G , we denote its center by $Z(G)$.

The homotopy groups of the suspended spaces $\Sigma K(A, 1)$ are highly nontrivial from the point of view of computations. For example, the homotopy groups $\pi_n(\Sigma K(\mathbb{Z}/2, 1))$ are known at the moment for $n \leq 6$ (see [11], [14]).

Our construction is as follows. Let A be an abelian group. For $n \geq 1$, define the free product $T_n := (\underbrace{A \times \cdots \times A}_{n \text{ copies}}) * (\underbrace{A \times \cdots \times A}_{n \text{ copies}})$. For $i = 1, \dots, n$, denote by

A_i (resp. A_{n+i}) the i th copy of A in the first (resp. second) free summand in T_n . For $j = 1, \dots, 2n$, $a \in A$ we denote by a_j its value in A_j . Define

$$(1.1) \quad R_1 = \langle a_1, a_{n+1} \mid a \in A \rangle^{T_n}$$

$$(1.2) \quad R_i = \langle a_i a_{i-1}^{-1}, a_{n+i} a_{n+i-1}^{-1} \mid a \in A \rangle^{T_n}, \quad 1 < i \leq n$$

$$(1.3) \quad R_{n+1} = \langle a_n, a_{2n} \mid a \in A \rangle^{T_n}$$

Recall the definition of the symmetric commutator subgroup:

$$(1.4) \quad [R_1, R_2, \dots, R_{n+1}]_S = \prod_{\sigma \in \Sigma_{n+1}} [\dots [R_{\sigma(1)}, R_{\sigma(2)}], \dots, R_{\sigma(n+1)}].$$

Define

$$\mathcal{J}_n(A) := T_n / (\gamma_{2^{n+1}}([T_n, T_n])[R_1, \dots, R_{n+1}]_S),$$

where, for a group G , $\{\gamma_i(G)\}_{i \geq 1}$ is the lower central sequence. The main result of the paper is the following:

Theorem 1.1. *Let A be a finite abelian group. The homotopy group $\pi_{n+1}(\Sigma K(A, 1))$ is isomorphic to the center of the polycyclic group $\mathcal{J}_n(A)$ for all $n \geq 2$.*

All polycyclic groups are finitely presented, in particular, $\mathcal{J}_n(A)$ is finitely presented, moreover, it follows from the definition, that $\mathcal{J}_n(A)$ is virtually nilpotent. The group $\mathcal{J}_n(A)$ is obtained canonically from the simplicial group $K(A, 1) * K(A, 1)$. The lower central series spectral sequence from the free product $K(A, 1) * K(A, 1)$ may give some computational information on the homotopy groups and the group $\mathcal{J}_n(A)$.

There is an explicit construction of finitely presented group whose center is $\pi_n(S^2) \times \mathbb{Z}$ given in [7]. However there are some mistakes in the paper [7]. In Section 4, we give a brief review for the main result in [7] with corrections for the mistakes. The notion of symmetric commutator (1.4) plays the central role in this construction.

2. SIMPLICIAL MODELS

Let A be an abelian group. The homotopy commutative diagram of fibre sequences

$$(2.1) \quad \begin{array}{ccccc} \Sigma K(A, 1) \wedge K(A, 1) & \xrightarrow{H} & \Sigma K(A, 1) & \longrightarrow & K(A, 2) = BK(A, 1) \\ \parallel & & \downarrow & \text{pull} & \downarrow \Delta \\ \Sigma K(A, 1) \wedge K(A, 1) & \longrightarrow & K(A, 2) \vee K(A, 2) & \hookrightarrow & K(A, 2) \times K(A, 2), \end{array}$$

where H is the Hopf fibration, implies that there are isomorphisms

$$(2.2) \quad \pi_n(\Sigma K(A, 1)) \cong \pi_n(K(A, 2) \vee K(A, 2))$$

for $n \geq 3$.

Choose the simplest simplicial model for $K(A, 1)$. Applying the inverse to the normalization functor in the sense of Dold-Kan to the complex $A[1]$, we obtain the abelian simplicial group E_* with components

$$E_i = \underbrace{(A \times \cdots \times A)}_{i \text{ copies}}, \quad i \geq 1$$

and the property $|E_*| \simeq K(A, 1)$. The face and degeneracy maps in E_* are standard, their structure follows from the construction of the inverse to the normalization functor.

The following fact follows directly from the Whitehead Theorem [15] (see [6], Proposition 4.3, for the simplicial version of the Whitehead Theorem):

Lemma 2.1. *For an abelian group A , there is a homotopy equivalence*

$$|E_* * E_*| \simeq \Omega(K(A, 2) \vee K(A, 2)).$$

Here $E_* * E_*$ is the free product of two copies of the simplicial group E_* .

Recall that a simplicial group G_* is called free if each G_i , $i \geq 0$ is free with a given basis, and the bases are stable under degeneracy operations. The following result is due to Curtis [4]

Theorem 2.2. *Let G_* be a connected free simplicial group, then for each $r \geq 2$, the homomorphism of simplicial groups $G \rightarrow G/\gamma_r(G)$ induces isomorphisms*

$$\pi_i(G) \simeq \pi_i(G/\gamma_r(G))$$

for all $i < \log_2 r$.

Now we consider the free product of simplicial groups $T_* := E_* * E_*$. It has the following components

$$T_i := \underbrace{(A \times \cdots \times A)}_{i \text{ copies}} * \underbrace{(A \times \cdots \times A)}_{i \text{ copies}}, \quad i \geq 1.$$

Lemma 2.3. *The simplicial group $[T_*, T_*]$ is free.*

Proof. For abelian groups B, C , the commutator subgroup $[G, G]$ of the free product $G = B * C$, is a free group with basis given by all commutators² $[b, c]$, $1 \neq b \in B$, $1 \neq c \in C$ (see, for example, [9]). Taking such commutators as basis elements in $[T_*, T_*]$, we immediately obtain from definition of T_* , that the bases are stable under degeneracy operations. \square

Now Theorem 2.2 and Lemma 2.3 imply that, the natural map of simplicial groups

$$[T_*, T_*] \rightarrow [T_*, T_*]/\gamma_{2^{n+1}}([T_*, T_*])$$

induces isomorphism of homotopy groups

$$\pi_i([T_*, T_*]) \simeq \pi_i([T_*, T_*]/\gamma_{2^{n+1}}([T_*, T_*])), \quad i < n.$$

²We use the standard notation $[a, b] := a^{-1}b^{-1}ab$.

The short exact sequence of simplicial groups:

$$1 \rightarrow [T_*, T_*] \rightarrow T_* \rightarrow T_*/[T_*, T_*] \rightarrow 1$$

induces the long exact sequence of homotopy groups. The homotopy equivalence

$$|T_*/[T_*, T_*]| \simeq K(A, 1) \times K(A, 1)$$

implies that, for $i \geq 3$, there is an isomorphism

$$\pi_i([T_*, T_*]) \simeq \pi_i(T_*).$$

Lemma 2.1 together with isomorphisms (2.2) imply that, for all $2 < i < n$, there are isomorphisms of homotopy groups

$$(2.3) \quad \pi_i(T_*) \simeq \pi_i(T_*/\gamma_{2^{n+1}}([T_*, T_*])) \simeq \pi_{i+1}(\Sigma K(A, 1)).$$

3. PROOF OF THE MAIN THEOREM

Lemma 3.1. *Let A and B be nontrivial finite abelian groups, $G = A * B$. The center of the group $H = G/\gamma_n([G, G])$ is trivial for $n \geq 2$.*

Proof. The commutator subgroup $[G, G]$ is free with a basis $\{[a, b], 1 \neq a \in A, 1 \neq b \in B\}$ (see [9]). We prove the statement for $n = 2$, the proof for higher n is similar.

Let $h \in [G, G]$ then, modulo $\gamma_2([G, G])$, h can be uniquely written as follows

$$(3.1) \quad h = \prod_{1 \neq a \in A, 1 \neq b \in B} [a, b]^{m(a,b)}, \quad m(a,b) \in \mathbb{Z}.$$

For $c \in A$, we have

$$h^c \equiv \prod_{1 \neq a \in A, 1 \neq b \in B} [ca, b]^{m(a,b)} [c, b]^{-m(a,b)} \pmod{\gamma_2([G, G])}$$

and

$$[h, c] \equiv \prod_{1 \neq a \in A, 1 \neq b \in B} [ca, b]^{m(a,b)} [a, b]^{-m(a,b)} [c, b]^{-m(a,b)} \pmod{\gamma_2([G, G])}$$

Assume that $1 \neq \alpha \in Z(H)$. We can write α as $\alpha = f d h \cdot \gamma_2([G, G])$, $f \in A$, $d \in B$, $h \in [G, G]$. Assume that $d \neq 1$. Writing α in the form (3.1) and taking $c \in A$, we have

$$[c, \alpha] \equiv [c, d] \prod_{1 \neq a \in A, 1 \neq b \in B} [ca, b]^{m(a,b)} [a, b]^{-m(a,b)} [c, b]^{-m(a,b)} \pmod{\gamma_2([G, G])}$$

Since $\alpha \in Z(H)$, and $ca \neq c$ for $a \neq 1$, we have

$$(3.2) \quad m(c, d) + \sum_{1 \neq a \in A} m(a, d) = 1.$$

Since we can choose arbitrary element $c \in A$, the identity (3.2) holds for every $c \in A$ (but $d \in B$ is fixed). Summing up over all $c \in A$, we have

$$(1 + |A|) \left(\sum_{1 \neq a \in A} m(a, d) \right) = |A|$$

but this is not possible, since all coefficients $m(a, d) \in \mathbb{Z}$. Therefore, $d = 1$. Analogous argument shows that $f = 1$ and hence $\alpha \in [G, G] \cdot \gamma_2([G, G])$. Now we present $\alpha = h \cdot \gamma_2([G, G])$ in the form (3.1). Since $\alpha \in Z(H)$, we have

$$[h, c] \equiv \prod_{1 \neq a \in A, 1 \neq b \in B} [ca, b]^{m(a,b)} [a, b]^{-m(a,b)} [c, b]^{-m(a,b)} \equiv 0 \pmod{\gamma_2([G, G])}$$

for any $c \in A$. Therefore, for every $b \in B$, we have

$$(3.3) \quad m(c, b) + \sum_{1 \neq a \in A} m(a, b) = 0.$$

Again, summing up over all $c \in A$, we have

$$(1 + |A|) \sum_{1 \neq a \in A} m(a, b) = 0$$

and therefore, $\sum_{1 \neq a \in A} m(a, b) = 0$. Now (3.3) implies that $m(c, b) = 0$ for all $c \in A$, $b \in B$. Therefore, $h \in \gamma_2([G, G])$ and the statement is proved. \square

Proof of Theorem 1.1. Lemma 3.1 implies that, the centers of the components of the simplicial group $T_*/\gamma_{2n+1}([T_*, T_*])$ are trivial. Isomorphism (2.3) together with [16, Proposition 2.14] imply that there is an isomorphism

$$\pi_{n+1}(\Sigma K(A, 1)) \simeq Z(T_*/\gamma_{2n+1}([T_*, T_*])\mathcal{B}_n)$$

where \mathcal{B}_n is the Moore boundary. It remains to show that the Moore boundary is given by symmetric commutator subgroup

$$(3.4) \quad \mathcal{B}_n = [R_1, R_2, \dots, R_{n+1}]_S.$$

Let $F = F(A \setminus \{1\})$ be the free group free generated by all nontrivial elements of A^3 . Let $\phi: F \rightarrow A$ be the canonical quotient homomorphism, namely ϕ is the (unique) group homomorphism such that $\phi(a) = a$ for $a \in A \setminus \{1\}$. Then ϕ induces a simplicial epimorphism⁴

$$\tilde{\phi}: F^F[S^1] \twoheadrightarrow F^A[S^1] \twoheadrightarrow F^A[S^1]^{\text{ab}} = K(A, 1).$$

Recall that the simplicial circle S^1 has the elements that can be explicitly given by

$$S_n^1 = \{*, x_{i+1} = s_{n-1} \cdots s_{i+1} s_i \cdots s_0 \sigma_1 \mid 0 \leq i \leq n-1\},$$

where σ_1 is the nondegenerate element in S_1^1 , and by the definition of Carlsson's construction [3], $F^F[S^1]_n$ is the self free product of F index by elements in $S_n^1 \setminus \{*\}$. Thus

$$F^F[S^1]_n = (F)_{x_1} * (F)_{x_2} * \cdots * (F)_{x_n},$$

where $(F)_{x_i}$ is a copy of F labeled by x_i . The epimorphism $\tilde{\phi}: F^F[S^1]_n \rightarrow K(A, 1)_n$ is explicitly given by the composite

$$(F)_{x_1} * (F)_{x_2} * \cdots * (F)_{x_n} \twoheadrightarrow (A)_{x_1} * (A)_{x_2} * \cdots * (A)_{x_n} \twoheadrightarrow (A)_{x_1} \times (A)_{x_2} \times \cdots \times (A)_{x_n}.$$

Thus

$$\tilde{\phi}((a)_{x_i}) = a_i$$

for $a \in A \setminus \{0\}$ for $1 \leq i \leq n$. Consider the epimorphism

$$\tilde{\phi} * \tilde{\phi}: F^F[S^1] * F^F[S^1] \twoheadrightarrow K(A, 1) * K(A, 1).$$

³Here we use the multiplicative notations for the elements of abelian groups.

⁴For the description of Carlsson's construction $F^G[S^1]$, see [3], [16].

Observe that $F^F[S^1] * F^F[S^1] = F^{F*F}[S^1]$. Following the notation in the introduction for the group T_n , let B be a copy of A and so the group $F * F$ is the free group with a basis given by a for $a \in A \setminus \{1\}$ and b for $b \in B \setminus \{1\}$. The epimorphism

$$\tilde{\phi} * \tilde{\phi}: F^{F*F}[S^1]_n \rightarrow K(A, 1)_n * K(A, 1)_n = T_n$$

is given by

$$(\tilde{\phi} * \tilde{\phi})((a)_{x_i}) = a \in A_i \text{ and } (\tilde{\phi} * \tilde{\phi})((b)_{x_i}) = b \in B_i$$

for $1 \leq i \leq n$. Let

$$\begin{aligned} \tilde{R}_1 &= \langle (a)_{x_1}, (b)_{x_1} \mid a \in A \setminus \{1\}, b \in B \setminus \{1\} \rangle^{F^{F*F}[S^1]_n} \\ \tilde{R}_i &= \langle (a)_{x_i} (a)_{x_{i-1}}^{-1}, (b)_{x_i} (b)_{x_{i-1}}^{-1} \mid a \in A \setminus \{1\}, b \in B \setminus \{1\} \rangle^{F^{F*F}[S^1]_n}, \quad 1 < i \leq n \\ \tilde{R}_{n+1} &= \langle (a)_{x_n}, (b)_{x_n} \mid a \in A \setminus \{1\}, b \in B \setminus \{1\} \rangle^{F^{F*F}[S^1]_n}. \end{aligned}$$

Then

$$\tilde{\phi} * \tilde{\phi}(\tilde{R}_i) = R_i$$

for $1 \leq i \leq n$ and so

$$(3.5) \quad \tilde{\phi} * \tilde{\phi}([\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_{n+1}]_S) = [R_1, R_2, \dots, R_{n+1}]_S.$$

Let H_j be a sequence of subgroups of G for $1 \leq j \leq k$. Recall that the fat commutator subgroup $[[H_1, \dots, H_k]]$ of G is generated by all of the commutators $\beta^t(h_{i_1}^{(1)}, \dots, h_{i_t}^{(t)})$, where

- 1) $1 \leq i_s \leq k$;
- 2) all integers in $\{1, 2, \dots, k\}$ appear as at least one of the integers i_s ;
- 3) $h_j^{(s)} \in H_j$;
- 4) for each $t \geq k$, β^t runs over all of the bracket arrangements of weight t .

By [16, Proof of Theorem 1.8], the Moore boundary

$$\mathcal{B}_n F^{F*F}[S^1] = [[\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_{n+1}]].$$

According to [8, Theorem 1.1],

$$[[\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_{n+1}]] = [\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_{n+1}]_S$$

It follows that

$$(3.6) \quad \mathcal{B}_n F^{F*F}[S^1] = [\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_{n+1}]_S.$$

By [13, Lemma 5, 3.8], any simplicial epimorphism induces an epimorphism on the Moore chains and so

$$\begin{aligned} \mathcal{B}_n(K(A, 1) * K(A, 1)) &= d_0 N_{n+1}(K(A, 1) * K(A, 1)) \\ &= d_0(\tilde{\phi} * \tilde{\phi}(N_{n+1}(F^{F*F}[S^1]))) \\ &= \tilde{\phi} * \tilde{\phi}(d_0 N_{n+1}(F^{F*F}[S^1])) \\ &= \tilde{\phi} * \tilde{\phi}(\mathcal{B}_n F^{F*F}[S^1]). \end{aligned}$$

Equation (3.4) follows from equations (3.5) and (3.6) now. This finishes the proof. Observe that, for every pair A, B of finite abelian groups, the quotient group $A * B / \gamma_n([A * B, A * B])$ is polycyclic, i.e. it is a solvable group such that every its subgroup is finitely-generated. In particular, the groups $\mathcal{J}_n(A) = T_n / \gamma_{2^{n+1}}([T_n, T_n]) \mathcal{B}_n$ are finitely presented for all $n \geq 2$. \square

4. HOMOTOPY GROUPS OF S^2

Taking the simplicial group $E_* * E_*$ from Section 2, for $A = \mathbb{Z}$, we get the homotopy equivalence

$$|E_* * E_*| \simeq \Omega(K(\mathbb{Z}, 2) \vee K(\mathbb{Z}, 2))$$

Since the Moore boundaries of $E_* * E_*$ can be described combinatorially using symmetric commutators, we obtain the following description of homotopy groups of S^2 alternative to the description given in [16].

Proposition 4.1. *Let $n \geq 3$, $T_n := \underbrace{(\mathbb{Z} \times \cdots \times \mathbb{Z})}_{n \text{ copies}} * \underbrace{(\mathbb{Z} \times \cdots \times \mathbb{Z})}_{n \text{ copies}}$. There is an isomorphism*

$$\pi_{n+1}(S^2) \simeq Z(T_n/[R_1, \dots, R_{n+1}]_S),$$

where the subgroups R_i , $i = 1, \dots, n+1$ are defined as in (1.1)-(1.3).

Observe that, the groups $T_n/[R_1, \dots, R_{n+1}]_S$ are not finitely presented as well as the groups from [16] whose center is $\pi_{n+1}(S^2)$. There is an explicit construction of finitely presented group whose center is $\pi_n(S^2) \times \mathbb{Z}$ given in [7]. However there are some mistakes in the paper [7]. In this section, we give a brief review for the main result in [7] with corrections for the mistakes.

Let $d_i: P_n \rightarrow P_{n-1}$ be the group homomorphism given by removing the i th strand of n -strand pure braids for $1 \leq i \leq n$. There exists a well-defined additional face operation $d_0: P_n \rightarrow P_{n-1}$ as a group homomorphism defined by

$$d_0 A_{i,j} = A_{i-1,j-1}$$

for $1 \leq i < j \leq n$, where the braid $A_{0,j} \in P_{n-1}$ is given by

$$\begin{aligned} A_{0,j} &= (A_{j,j+1} A_{j,j+2} \cdots A_{j,n-1})^{-1} (A_{1,j} \cdots A_{j-1,j})^{-1} \\ &= (\sigma_j \cdots \sigma_{n-3} \sigma_{n-2}^2 \sigma_{n-3} \cdots \sigma_j)^{-1} \cdot (\sigma_{j-1} \cdots \sigma_2 \sigma_1^2 \sigma_2 \cdots \sigma_{j-1})^{-1}. \end{aligned}$$

Then the sequence of groups $\mathbb{P} = \{P_n\}_{n \geq 0}$ forms a Δ -group with the property that [7, Proposition 2.5] the Moore homotopy group of the Δ -group \mathbb{P} given by

$$(4.1) \quad \pi_n(\mathbb{P}) \cong \pi_n(S^2).$$

From the definition of the Moore homotopy groups of Δ -groups, the Moore chains

$$(4.2) \quad N_n \mathbb{P} = \bigcap_{i=1}^n \text{Ker}(d_i: P_n \rightarrow P_{n-1}) = \text{Brun}_n,$$

where Brun_n is the group of n -strand pure Brunnian braids. Let

$$\text{Bd}_n = d_0(\text{Brun}_{n+1}) = \mathcal{B}_n \mathbb{P}$$

be the Moore boundaries of \mathbb{P} , which is called n -strand boundary Brunnian braids in [7]. The main result of [7] is as follows.

Theorem 4.2. [7, Theorem 1.1 (2)] *The group Bd_n is a normal subgroup of the Artin braid group B_n . There are isomorphisms of groups*

$$Z(P_n/\text{Bd}_n) \cong \pi_n(S^2) \times \mathbb{Z} \text{ and } Z(B_n/\text{Bd}_n) \cong \{\alpha \in \pi_n(S^2) \mid 2\alpha = 0\} \times \mathbb{Z}$$

for $n \geq 4$, where \mathbb{Z} is induced from $Z(P_n) = Z(B_n) = \mathbb{Z}$. \square

The main results [7, Theorem 1, Theorem 3] are correct. Also it is correct that the groups P_n/Bd_n and B_n/Bd_n are finitely presented. The major mistake in [7] is that [7, Lemma 3.6] is not correct. As a consequence of this false lemma, the statements [7, Theorem 3.7, Lemma 3.11, Corollary 3.12, Corollary 3.13, Corollary 3.14, Proposition 3.15] are not true. For the introduction to the main results in [7], the set of normal generators for Brun_n in [7, line 3, p.522] and for Bd_n in [7, line 14, p.523] are not correct. In order to correct the mistakes, we determine a finite set of normal generators for Brun_n as well as a finite set of normal generators for Bd_n . This will confirm that P_n/Bd_n and B_n/Bd_n are finitely presented.

Theorem 4.3. *Let Brun_n be the group of n -strand Brunnian pure braids and let Bd_n be the group of n -strand boundary Brunnian pure braids. Then*

- 1) *The group Brun_n is the normal closure of the following elements in P_n :*

$$[[[A_{1,n}, A_{\sigma(2),j_2}], A_{\sigma(3),j_3}], \dots, A_{\sigma(n-1),j_{n-1}}]$$

for $\sigma \in \Sigma_{n-2}$ acting on $\{2, 3, \dots, n-1\}$ and $1 \leq j_2, \dots, j_{n-1} \leq n$ with $j_s \neq \sigma(s)$ for $2 \leq s \leq n-1$, where $A_{j,i} = A_{i,j}$ if $i > j$.

- 2) *The group Bd_n is the normal closure of the following elements in P_n :*

$$[[[A_{0,n}, A_{\sigma(1),j_1}], A_{\sigma(2),j_2}], \dots, A_{\sigma(n-1),j_{n-1}}]$$

for $\sigma \in \Sigma_{n-1}$ and $1 \leq j_1, \dots, j_{n-1} \leq n$ with $j_s \neq \sigma(s)$ for $2 \leq s \leq n-1$, where $A_{j,i} = A_{i,j}$ if $i > j$.

Proof. (1). Let $R_{i,j}$ be the normal closure of $A_{i,j}$ in P_n for $1 \leq i < j \leq n$. From [1, Theorem 1.1],

$$\text{Brun}_n = \prod_{\sigma \in \Sigma_{n-1}} [[R_{\sigma(1),n}, R_{\sigma(2),n}], \dots, R_{\sigma(n-1),n}].$$

By [8, Theorem 1.2],

$$\prod_{\sigma \in \Sigma_{n-1}} [[R_{\sigma(1),n}, R_{\sigma(2),n}], \dots, R_{\sigma(n-1),n}] = \prod_{\sigma \in \Sigma_{n-2}} [[R_{1,n}, R_{\sigma(2),n}], \dots, R_{\sigma(n-1),n}].$$

Thus

$$(4.3) \quad \text{Brun}_n = \prod_{\sigma \in \Sigma_{n-2}} [[R_{1,n}, R_{\sigma(2),n}], \dots, R_{\sigma(n-1),n}].$$

For each $1 \leq i \leq n$, let G_i be the subgroup of P_n generated by $A_{i,j}$ for $1 \leq j \leq n$ with $j \neq i$, where $A_{i,j} = A_{j,i}$ if $i > j$. Recall that $P_n = \pi_1(F(\mathbb{C}, n))$, where

$$(4.4) \quad F(\mathbb{C}, n) = \{(z_1, \dots, z_n) \mid z_i \neq z_j \text{ for } i \neq j\}$$

is the ordered configuration space. By [1, Proposition 3.2], the removal of the i th strand operation $d_i: P_n \rightarrow P_{n-1}$ is induced by the coordinate projection

$$\pi_i: F(\mathbb{R}^2, n) \longrightarrow F(\mathbb{R}^2, n-1) \quad (z_1, z_2, \dots, z_n) \mapsto (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n).$$

Let (p_1, \dots, p_n) be a base-point of $F(\mathbb{C}, n)$. Namely p_1, \dots, p_n are n distinct points in the plane \mathbb{R}^2 . By the classical Fadell-Neuwirth Theorem [5], the coordinate projection π_i is a fibration with a punctured plane $\mathbb{R}^2 \setminus \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n\}$ as a fibre. By taking the fundamental groups to the fibration (4.4), we obtain

$$\text{Ker}(d_i: P_n \rightarrow P_{n-1}) = G_i = F_{n-1}.$$

Moreover for $1 \leq i < j \leq n$

$$G_i \cap G_j = \text{Ker}(d_i: P_n \rightarrow P_{n-1}) \cap \text{Ker}(d_j: P_n \rightarrow P_{n-1}) = R_{i,j}$$

since the restriction of d_i to G_j is given by the projection map

$$F(A_{1,j}, \dots, A_{j-1,j}, A_{j,j+1}, \dots, A_{j,n}) \longrightarrow F(A_{1,j-1}, \dots, A_{j-1,n-1}) \leq P_{n-1}$$

with $d_i A_{i,j} = 1$, $d_i A_{s,j} = A_{s,j-1}$ for $s < i$. $d_i A_{s,j} = A_{s-1,j-1}$ for $i < s < j$ and $d_i A_{j,s} = A_{j-1,s-1}$ for $s > j$. Consider the factors in product (4.3). For each $\sigma \in \Sigma_{n-2}$,

$$[[R_{1,n}, R_{\sigma(2),n}], \dots, R_{\sigma(n-1),n}] \leq [[R_{1,n}, G_{\sigma(2)}], \dots, G_{\sigma(n-1)}].$$

Observe that

$$[[R_{1,n}, G_{\sigma(2)}], \dots, G_{\sigma(n-1)}] \leq \text{Brun}_n.$$

We can construct a finite set of normal generators in P_n for $[[R_{1,n}, G_{\sigma(2)}], \dots, G_{\sigma(n-1)}]$ based on the following statement:

Statement 4.4 [1, Proof of Lemma 5.2]. Let G be a group and let A and B be normal subgroups of G . If $\{a_i \mid i \in I\}$ is a set of normal generators for A in G and $\{b_j \mid j \in J\}$ is a set of generators for B , then $\{[a_i, b_j] \mid i \in I, j \in J\}$ is a set of normal generators for the commutator subgroup $[A, B]$ in G .

The construction of a set of normal generators for $[[R_{1,n}, G_{\sigma(2)}], \dots, G_{\sigma(n-1)}]$ is as follows. Observe that $R_{1,n}$ has a normal generator $A_{1,n}$ and $G_{\sigma(2)}$ has generators $A_{\sigma(2),j_2}$ for $1 \leq j_2 \leq n$ with $j_2 \neq \sigma(2)$. From the above statement, a set of normal generators for $[R_{1,n}, G_{\sigma(2)}]$ is given by

$$[A_{1,n}, A_{\sigma(2),j_2}]$$

for $1 \leq j_2 \leq n$ with $j_2 \neq \sigma(2)$. By repeating this procedure, a set of normal generators for $[[R_{1,n}, G_{\sigma(2)}], \dots, G_{\sigma(n-1)}]$ is given by

$$[[[A_{1,n}, A_{\sigma(2),j_2}], A_{\sigma(3),j_3}], \dots, A_{\sigma(n-1),j_{n-1}}]$$

for $1 \leq j_s \leq n$ with $j_s \neq \sigma(s)$. This finishes the proof of assertion (1).

(2). By definition, $\text{Bd}_n = d_0(\text{Brun}_{n+1})$. Since

$$d_0([A_{1,n+1}, A_{\sigma(1)+1,j_1+1}], \dots, A_{\sigma(n-1)+1,j_{n-1}+1}) = [[A_{0,n}, A_{\sigma(1),j_1}], \dots, A_{\sigma(n-1),j_{n-1}}]$$

with $[A_{1,n+1}, A_{\sigma(1)+1,j_1+1}], \dots, A_{\sigma(n-1)+1,j_{n-1}+1} \in \text{Brun}_{n+1}$, the elements listed in assertion (2) lie in Bd_n . Now from equation 4.3, we have

$$\begin{aligned} \text{Bd}_n &= \prod_{\sigma \in \Sigma_{n-1}} [[d_0(R_{1,n+1}), d_0(R_{\sigma(2),n+1})], \dots, d_0(R_{\sigma(n),n+1})] \\ (4.5) \quad &= \prod_{\sigma \in \Sigma_{n-1}} [[R_{0,n}, R_{\sigma(2)-1,n}], \dots, R_{\sigma(n)-1,n}], \end{aligned}$$

where $R_{0,j}$ is the normal closure of $A_{0,j}$ in P_n . Given any $\sigma \in \Sigma_{n-1}$ acting on $\{2, \dots, n\}$, the factor

$$[[R_{0,n}, R_{\sigma(2)-1,n}], \dots, R_{\sigma(n)-1,n}] \leq [[R_{0,n}, G_{\sigma(2)-1}], \dots, G_{\sigma(n)-1}]$$

with $1 \leq \sigma(t) - 1 \leq n - 1$ for $2 \leq t \leq n$. From Statement 4.4, a set of normal generators for $[[R_{0,n}, G_{\sigma(2)-1}], \dots, G_{\sigma(n)-1}]$ is given by

$$[[[A_{0,n}, A_{\sigma(2)-1,j_2}], A_{\sigma(3)-1,j_3}], \dots, A_{\sigma(n)-1,j_n}]$$

with $1 \leq j_2, j_3, \dots, j_n \leq n$ and $j_s \neq \sigma(s) - 1$ for $2 \leq s \leq n$. Assertion (2) follows. \square

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